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The Lebesgue Decomposition for Lattices of Projection Operators

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In this note, we establish the Lebesgue decomposition for s-bounded vector valued additive functions defined on lattices of sets.

Strongly bounded (s-bounded) set functions were studied by C. E. Rickart in [22], where a Lebesgue decomposition was established. Then Rickart's result and results of the author [10] were extended in [11]. M. M. Rao also recognized the importance of s-boundedness [21], and several recent results [2–4, 13, 17] emphasize that importance.

Another recent trend is the development of integration theory with respect to lattices of sets in order to have basic tools with which to study problems for which appropriate mathematical models involve lattices; cf. [1, 5–8, 14–16, 18, 19].

Restricting measures to the sets in an algebra of sets corresponds to considering an algebra of projection operators; this identification gives the Lebesgue decomposition for s-bounded additive functions defined on an algebra of sets as a consequence of the decomposition with respect to an algebra of projection operators that was established in [10]. The corresponding restrictions of set functions to elements of a lattice of sets correspond to a lattice of projection operators. Consequently, our decomposition with respect to lattices of projection operators yields the Lebesgue decomposition for s-bounded additive functions defined on a lattice of sets as a special case.

B. J. Pettis showed [20] that a set function λ defined on a lattice $\mathscr{M}$ of sets has an additive extension to the generated ring if, and only if, λ is modular: $\lambda(\phi) = 0$ and $\lambda(E \cup F) + \lambda(E \cap F) = \lambda(E) + \lambda(F)$, for all E and F in $\mathscr{M}$. A modular (additive) function λ is said to be s-bounded if the extension of λ to the generated ring is s-bounded.

The reader may wish to refer to [10] and [11] for a more leisurely introduction to the setting of the theorem than that which follows.

Let G be a generalized complete normed abelian group, where generalized means

(1) $\| 0 \| = 0$,

(2) if $x \neq 0$, then $0 < \| x \| \leqslant \infty$, and

(3) only the subgroup $\{x; \| x \| < \infty\}$ of bounded elements need be complete.

Let $\mathscr{U}$ be a Boolean algebra of projection operators on G such that if each of A and B is an element of $\mathscr{U}$, $A \leqslant B$, and $g \in G$, then $\| Ag \| \leqslant \| Bg \|$.

An element f of G is said to be s-bounded [22] if whenever $\{A_i\}$ is a sequence of pairwise disjoint elements of $\mathscr{U}$ it follows that $\lim_i A_i f = 0$.

Let $\mathscr{M}$ be a lattice of elements of $\mathscr{U}$ such that $\mathscr{M}$ contains the zero of $\mathscr{U}$.

Suppose that for each positive number x, $\mathscr{M}_x$ is a nonempty subset of $\mathscr{M}$ such that

(4) if $A \in \mathscr{M}$ and $A_x \in \mathscr{M}_x$, then $A \wedge A_x \in \mathscr{M}_x$, and

(5) if $A_x \in \mathscr{M}_x$ and $A_y \in \mathscr{M}_y$, then $A_x \vee A_y \in \mathscr{M}_{x+y}$.

For $g \in G$, let $S(g) = \lim_{x \to 0} (\sup A \in \mathscr{M}_x \| Ag \|)$.

THEOREM. *Suppose that f is a bounded and* s-*bounded element of G. Then there exist, uniquely, elements h and s of G such that*

(6) $f = h + s$,

(7) $S(h) = 0$, *and*

(8) *if $\epsilon > 0$, then there exists $A \in \mathscr{M}_\epsilon$ such that $\| A's \| < \epsilon$, where A' denotes the complement of A in $\mathscr{U}$.*

The theorem was established in [11] for the case $\mathscr{M} = \mathscr{U}$, and suitable modification of the argument given there will establish the general case. However, this decomposition may split f very differently from the way in which that decomposition splits the extension of f. If we think of a lattice of sets as corresponding to a cone of functions, the generated algebra as corresponding to the linear space of functions generated by the cone, and the absolutely continuous parts as the projections of a function on the cone and subspace respectively, then this latter contrast is to be expected. The following very elementary example makes the contrast between the decomposition for lattices and the decomposition for the generated algebras transparent.

Suppose $\mathscr{M}$ is comprised of sets $[0, x)$, where $0 \leqslant x \leqslant 1$, and μ is the restriction of Lebesgue measure to $\mathscr{M}$. Then any continuous (i.e., nonatomic) λ is absolutely continuous with respect to μ on $\mathscr{M}$ even though the extension of λ might be singular with respect to the extension of μ. Notice that the cone of $\mathscr{M}$-measurable (i.e., $[\phi > a] \in \mathscr{M}$, $a \in R$) real valued functions, ϕ, on $[0, 1]$ consists of the nonincreasing functions. Moreover, if ψ is a continuous function on $[0, 1]$ which is not of bounded variation and λ is defined by $\lambda([0, x)) = \psi(x)$, then λ is a bounded additive function on $\mathscr{M}$ whose extension to the generated algebra is not s-bounded.

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